Maximal Closed Set and Half-Space Separations in Finite Closure Systems

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Hyper-plane Separation in $\mathbb{R}^d$

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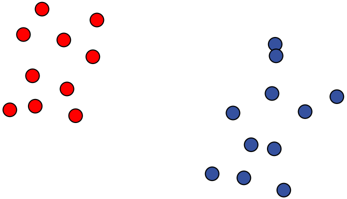

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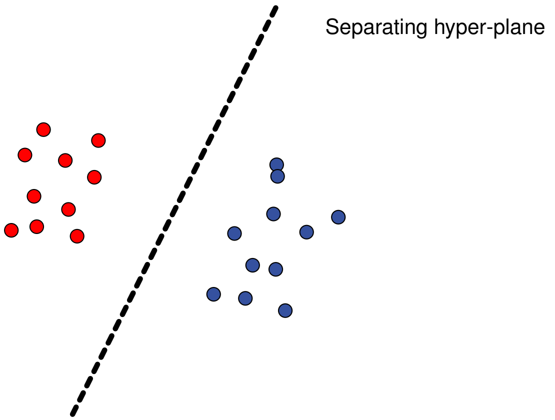

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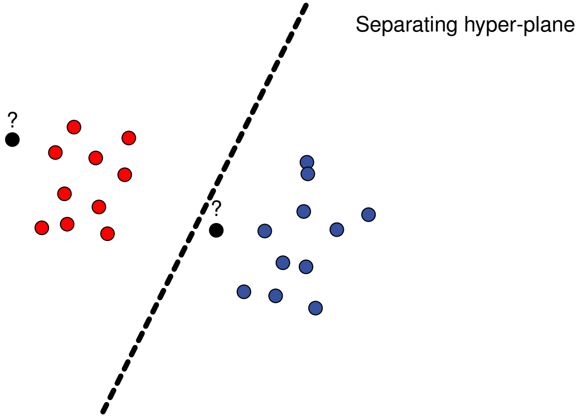

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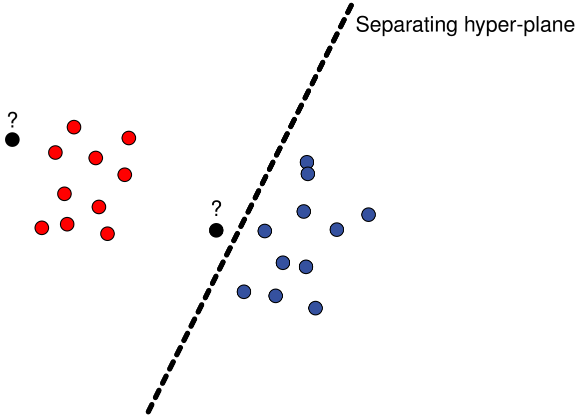

- classical machine learning problem (Rosenblatt, 1958), well-understood
- not unique

This Work: Generalization to Finite Set Systems

Definition:

- set system $(E, \mathcal{F})$: set E with $\mathcal{F} \subseteq 2^E$
- *finite* set system: $|E| < \infty$

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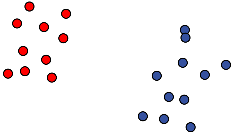

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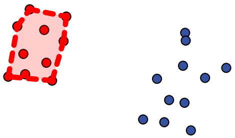

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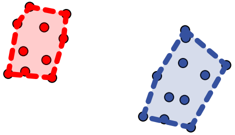

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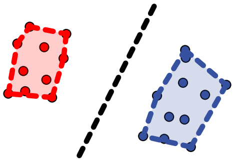

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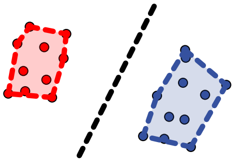

- practical:

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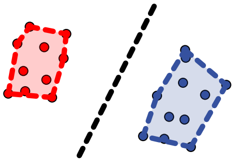

- practical: structured input space (examples in a few minutes)

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What is what:

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$\mathbb{R}^d$ $\rightarrow$

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What is what:

$\mathbb{R}^d \rightarrow E$ (ground set)

This Work: Generalization to Finite **Closure** Systems

What is what:

$\mathbb{R}^d$	$\rightarrow$	E (ground set)
convexity	$\rightarrow$	

This Work: Generalization to Finite **Closure** Systems

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$\mathbb{R}^d$	$\rightarrow$	E (ground set)
convexity	$\rightarrow$	closure operator

Definition:

closure operator c over E:

- $A \subseteq c(A)$
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Questions:

- separation algorithm
- classification algorithm

Some Practical Motivations:

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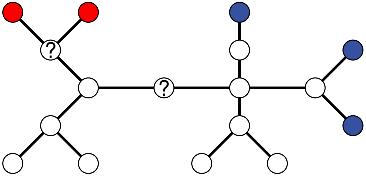
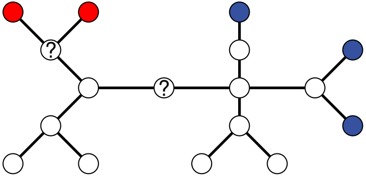


Trees

Some Practical Motivations: Graphs



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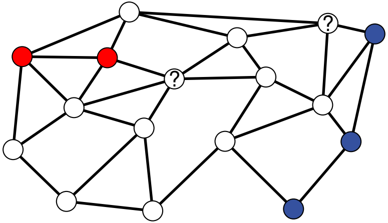

Graphs (e.g. molecule, social graph, . . .)

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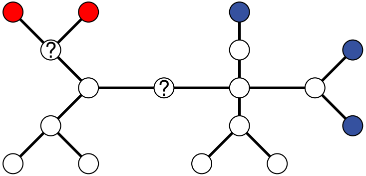

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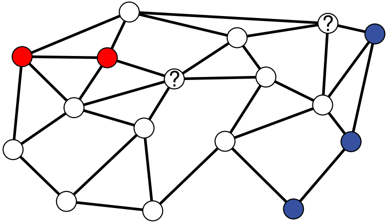

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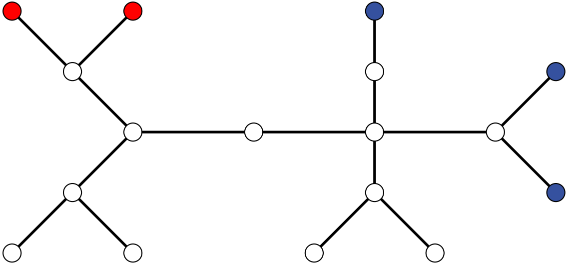

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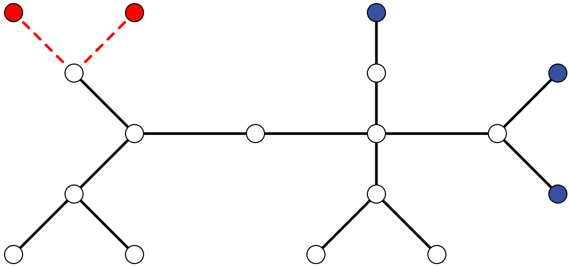

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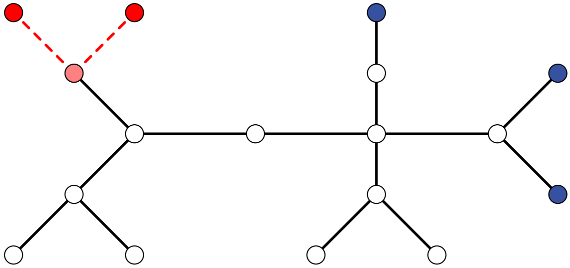

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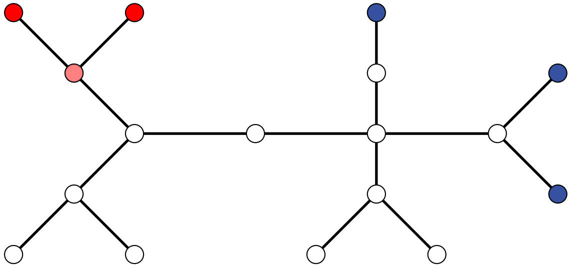

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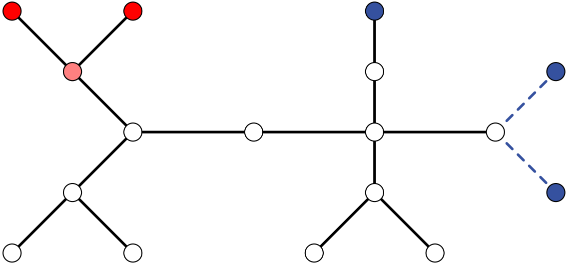

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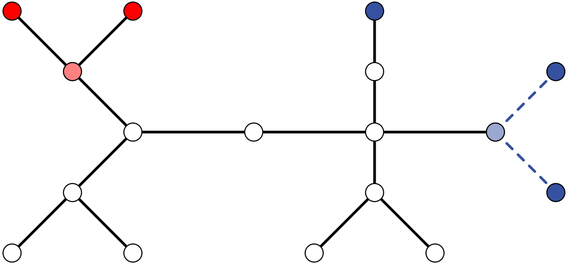

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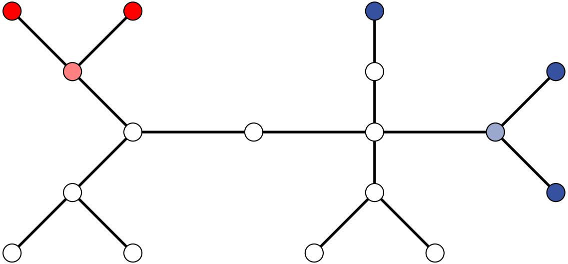

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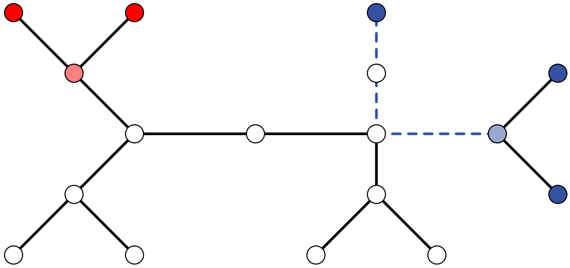

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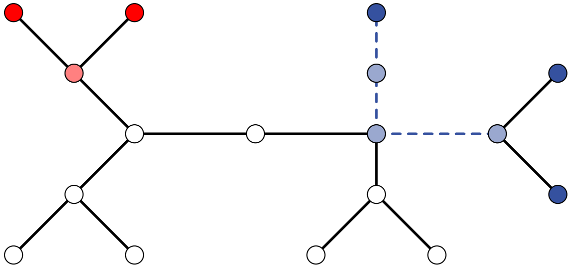

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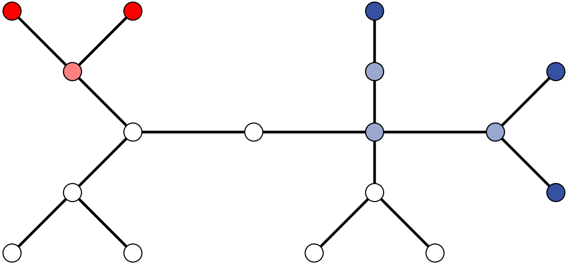
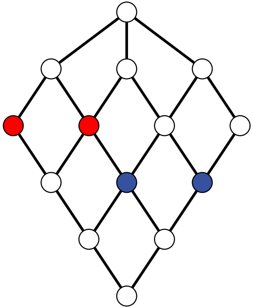


Some Practical Motivations: Lattices



Lattices (e.g. subsumption lattice, formal concept lattice, . . .)

Inductive Logic Programming (ILP) — Formal Concept Analysis (FCA)

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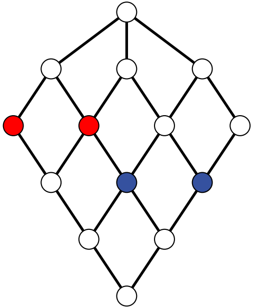

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formal concept lattice: Formal Concept Analysis (FCA)

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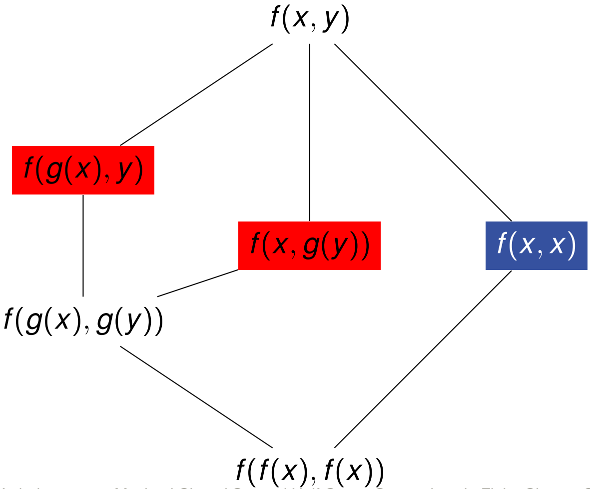

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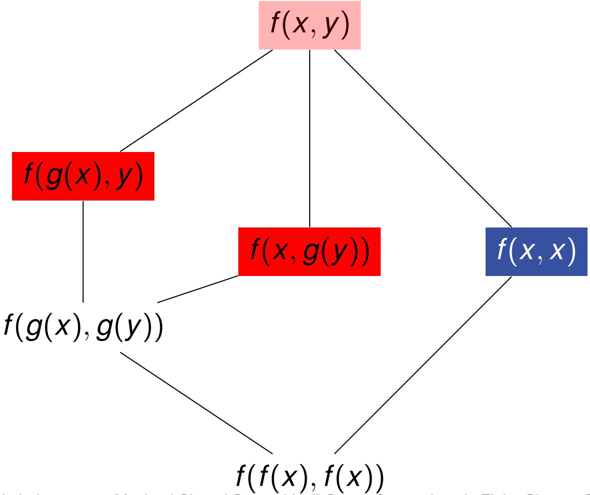

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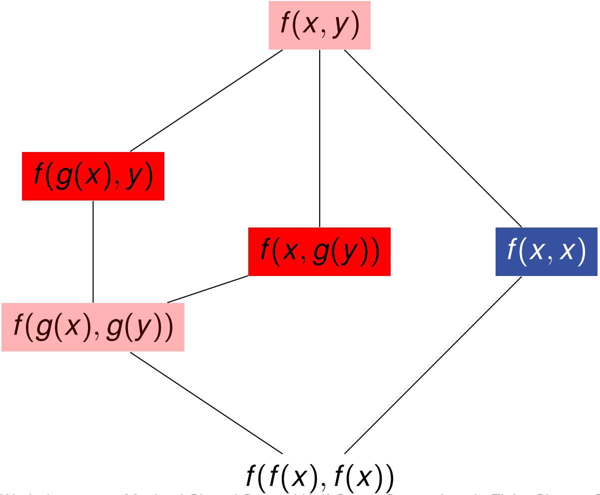

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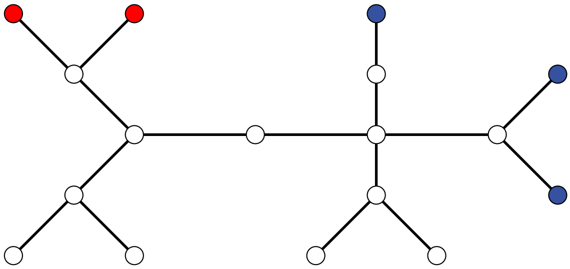

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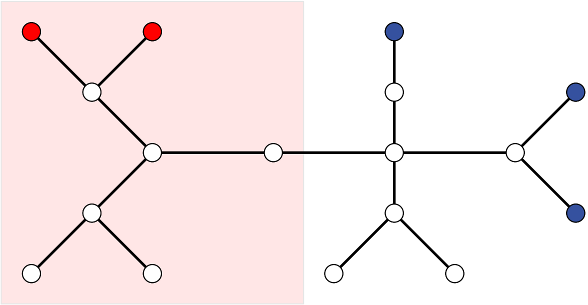

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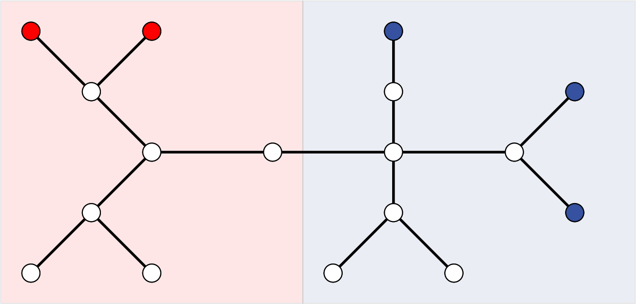

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- resort to Kakutani set systems $\rightarrow$ (set systems where Kakutani theorem holds)

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Given a closure system $(E, \mathcal{C})$ and sets $A, B \subseteq E$, find **maximal** disjoint closed sets separating A and B; print “No” if they do not exist

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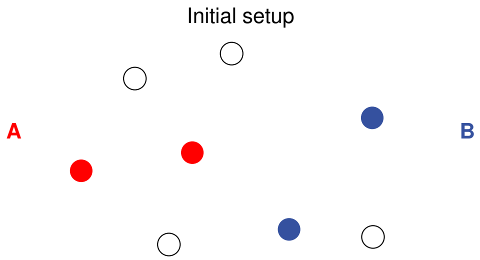

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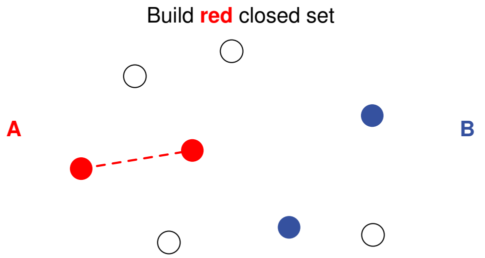

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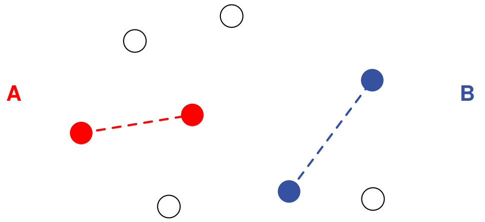
Build **blue** closed set



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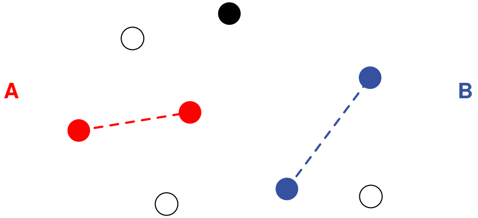
Take arbitrary point



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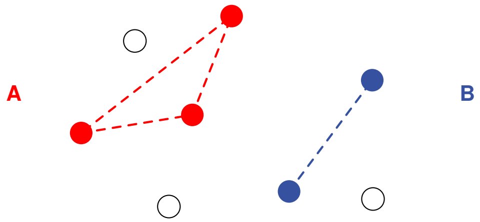

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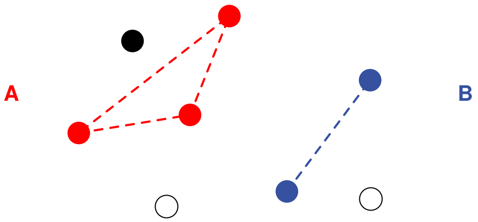

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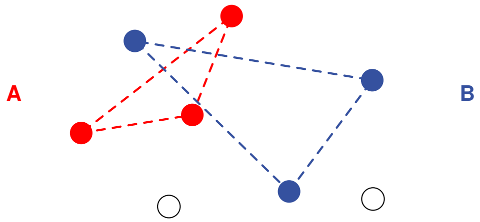

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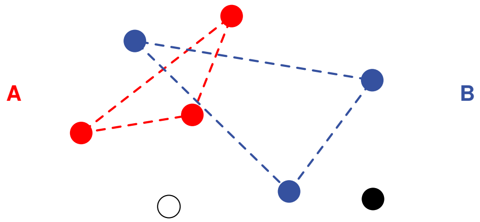

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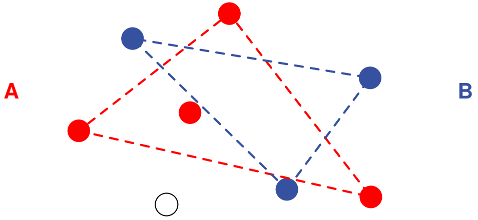

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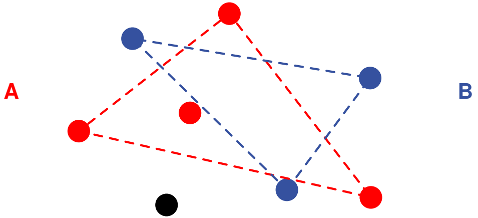

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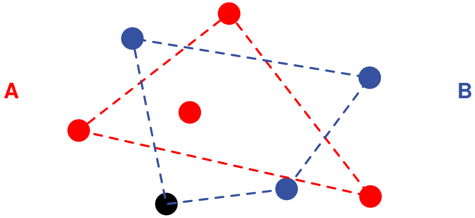

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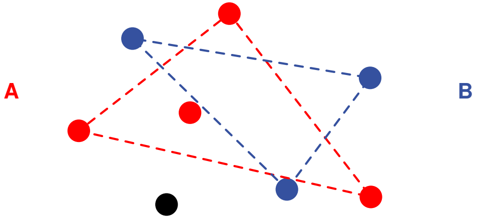

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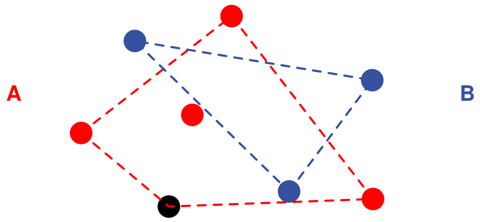

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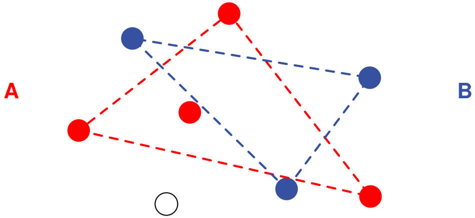

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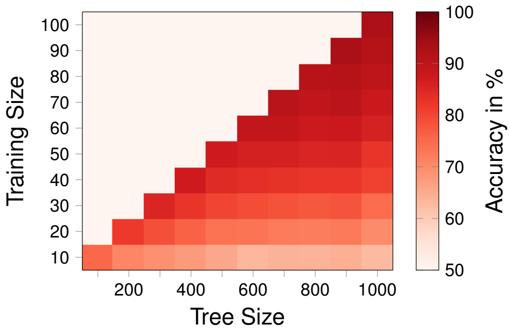
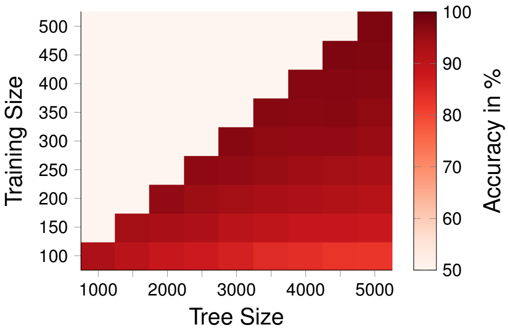
Experimental Results on Trees





- classification results of the maximal closed set separation algorithm on trees

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